Effect of Bearing Flexibility on Dual-Spin Satellite Attitude Stability

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The attitude stability of a symmetrical dual-spin satellite is investigated by use of the Liapunov method. The system reference frame is chosen such that the equations of motion are presented in a form for which a rather simple testing function can be constructed. Two stability criteria are obtained and as the system is completely damped, these stability conditions are necessary and sufficient. It is shown that one of the criteria is equivalent to the energy-sink criterion if the bearing stiffness of the system is sufficient to guarantee internal stability.

Introduction

THIS paper is an investigation of the attitude stability of a symmetric dual-spin satellite system with a flexible bearing. The bearing assembly is one of the most delicate parts of the system, as it has to transfer power, must operate almost without friction but still maintain a sufficient lateral stiffness. The flexibility of the bearing was the reason of the unexpected coning motion of TACSAT I.[1]

It will be assumed that the motions about the spin axes are controlled and that the corresponding angular rates are constant. Only the stability for the attitude and deformation variables will be analyzed. Energy dissipation will be associated with both parts of the bearing assembly and thus will introduce constraint damping forces.[2]

The problem was investigated in a previous paper[3] for an idealized configuration with energy dissipation in the platform only. In that case, the stability did not depend on the value of the damping coefficient and further the effect of gyroscopic coupling between the deformation variables could be ignored.

Equations of motion

The equations could be obtained by the use of the formalism for n-body systems.[4–6] However, the system is a deformable gyrostat, so that the formulation of Ref. 7 will be used. The equations of motion are the rotational equations of the whole system and the Lagrange equations for the deformation variables.

The linearized equations are derived, as asymptotic stability of the linearized system suffices to ensure asymptotic stability (that which is of interest) of the corresponding nonlinear system. The expression of the Lagrangian must thus include the quadratic terms in the variables.

At equilibrium, the relative angular momentum vector of the rotor is aligned with the nominal angular momentum of the system, **H**, and the principal axes of inertia coincide with the reference axes.

The reference frame $\{\hat{\mathbf{a}}_\alpha\}$ spins about axis $\hat{\mathbf{a}}_3$, aligned with **H**, at the nominal angular velocity of the platform, ω_0. The frames $\{\hat{\mathbf{x}}_\alpha\}$ and $\{\hat{\mathbf{x}}_\alpha'\}$ are associated with the platform and the rotor, respectively, with axes $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_3'$ being parallel to $\hat{\mathbf{a}}_3$ and the bearing axis when in equilibrium. Furthermore, we will define a body frame $\{\hat{\mathbf{x}}_\alpha\}$ centered at the center of mass of the system and coinciding with $\{\hat{\mathbf{a}}_\alpha\}$ when in equilibrium. During the motion, the orientation of $\{\hat{\mathbf{x}}_\alpha\}$ with respect to $\{\hat{\mathbf{a}}_\alpha\}$ will be described by the angles θ_1 θ_2 and θ_3, representing successive rotations about 1-, 2- and 3-axes and determining the intermediate frames $\{\hat{\mathbf{b}}_\alpha\}\{\hat{\mathbf{c}}_\alpha\}$. The angular velocity (with respect to inertial space) of $\{\hat{\mathbf{x}}_\alpha\}$ is then equal to

$$\boldsymbol{\omega} = \omega_0\hat{\mathbf{a}}_3 + \dot{\theta}_1\hat{\mathbf{a}}_1 + \dot{\theta}_2\hat{\mathbf{b}}_2 + \dot{\theta}_3\hat{\mathbf{c}}_3 = [\hat{\mathbf{x}}_\alpha]^T\omega$$

in which, for operational purpose, one defines a vector array $[\hat{\mathbf{x}}_\alpha] = [\hat{\mathbf{x}}_1\hat{\mathbf{x}}_2\hat{\mathbf{x}}_3]^T$ with vector elements. The elements of the matrix ω are then the components of the vector $\boldsymbol{\omega}$ in the frame $\{\hat{\mathbf{x}}_\alpha\}$ and are equal to

$$\omega_1 = \dot{\theta}_1 - \omega_0\theta_2, \quad \omega_2 = \dot{\theta}_2 + \omega_0\theta_1, \quad \omega_3 = \dot{\theta}_3 + \omega_0 \tag{1}$$

The orientation of the $\{\hat{\mathbf{x}}_\alpha\}$-frame with respect to $\{\hat{\mathbf{x}}_\alpha\}$ is described by three angles $\gamma_1, \gamma_2, \gamma_3$, representing rotations about 1-, 2-, and 3-axes, respectively. Similarly, the orientation of the $\{\hat{\mathbf{x}}_\alpha'\}$-frame with respect to $\{\hat{\mathbf{x}}_\alpha\}$ is described by the angles β_1, β_2, and ψ. Angles β_1 and β_2 describe the bearing deformation and determine the intermediate frame $\{\hat{\mathbf{y}}_\alpha\}$. The variable ψ is considered as cyclic and the corresponding angular velocity written $\boldsymbol{\Omega} = \Omega\hat{\mathbf{x}}_3'$ with Ω constant.

The kinematical relations between the vector bases are written under the form

$$\{\hat{\mathbf{x}}_\alpha\} = A[\hat{\mathbf{x}}_\alpha], \qquad [\hat{\mathbf{y}}_\alpha] = B[\hat{\mathbf{x}}_\alpha], \qquad [\hat{\mathbf{x}}_\alpha'] = C[\hat{\mathbf{y}}_\alpha]$$

where the matrix C is

$$C = \begin{bmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix A is a function of $\gamma_1, \gamma_2, \gamma_3$, which in quadratic approximation has the form

$$A = E - \tilde{\gamma} + 1/2\tilde{\gamma}\tilde{\gamma} + 1/2\bar{\gamma}U\bar{\gamma} \tag{2}$$

where E is the 3×3 identity matrix. Moreover, $\tilde{\gamma}$ is a skew-symmetric matrix defined as

$$\tilde{\gamma} = \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{bmatrix}$$

$\bar{\gamma}$ is the diagonal matrix

$$\bar{\gamma} = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix}$$

and U is the skew-symmetric matrix

$$U = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

It must be noted that quadratic form (2) is valid for any sequence of rotations about three different axes, but the form of matrix U may be different.[8]

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The expression of matrix B is obtained in a similar manner. We have just defined a total of eight variables describing a system with only five degrees of freedom, three rigid rotations and two internal degrees of freedom, so that we must introduce three relations between the variables. Note that we did not define the orientation of the frame $\{\hat{\mathbf{x}}_\alpha\}$ with respect to the body when deformation occurs, but only stated that in the equilibrium position this frame coincides with the reference frame. Let us define the orientation of $\{\hat{\mathbf{x}}_\alpha\}$ with respect to the platform, by the relation

$$\int \mathbf{r} \times \mathbf{u} dm = \mathbf{0} \tag{3}$$

where $\mathbf{r}$ is the position vector of the element of mass dm in the undeformed reference configuration of a similar body with frozen rotors ($\Omega = 0$) and $\mathbf{u}$ is the deformation vector of dm in this body, the value of the various variables being the same as in the actual system.

As all the variables are equal to zero in the equilibrium position, constant generalized forces will not appear in the equations and the Lagrange multipliers introduced to take the contraints into account are of the order of the other variables. It can then be concluded that the linearized system, obtained when the constraint equations have been linearized and solved, is equivalent to the corresponding linear system with Lagrange multipliers. Solving these constraints, the variables γ_1, γ_2, γ_3 are expressed in terms of β_1 and β_2 and the deformations can be given as functions of β_1 and β_2 only.

The time derivative of the total angular momentum of the body with respect to its center of mass is equal to the total torque applied about this point, or

$$\mathbf{H} = \mathbf{L} \tag{4}$$

where $\mathbf{H}$ is the total angular momentum with respect to the center of mass, and $\mathbf{L}$ is the resultant of external torques applied about the center of mass. By definition, $\mathbf{H}$ is the integral over the system of the moment of momentum, or

$$\mathbf{H} = \int \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} dm$$

where $\boldsymbol{\rho}$ is the position vector of dm with respect to the center of mass, dots denote time derivatives (with respect to some inertial space).

The vectors can be expressed in the "body-fixed" frame $\{\hat{\mathbf{x}}_\alpha\}$, in the form $\mathbf{H} = \{\hat{\mathbf{x}}_\alpha\}^T H$, $\boldsymbol{\rho} = [\hat{\mathbf{x}}_\alpha]^T \rho$, where the elements of the matrices H and ρ are the components of $\mathbf{H}$ and $\boldsymbol{\rho}$ in the vector basis $\{\hat{\mathbf{x}}_\alpha\}$.

The time derivative of $\boldsymbol{\rho}$ is

$$\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho} + \boldsymbol{\rho}^* = [\hat{\mathbf{x}}_\alpha]^T(\tilde{\omega}\rho + \dot{\rho}) \tag{5}$$

where $\boldsymbol{\rho}^*$ is defined as the vector with components equal to the time derivative of the elements of the matrix ρ or $\boldsymbol{\rho}^* = [\hat{\mathbf{x}}_\alpha]^T \dot{\rho}$, (this vector is referred to as the "relative velocity" vector of dm with respect to the frame $\{\hat{\mathbf{x}}_\alpha\}$) and the matrix

$$\tilde{\omega} = \begin{bmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The angular momentum vector can then be written in the form

$$\mathbf{H} = \int \boldsymbol{\rho} \times \boldsymbol{\rho}^* dm + \int \boldsymbol{\rho} \times \boldsymbol{\omega} \times \boldsymbol{\rho} dm$$

The inertia tensor of the body can be written as

$$\mathbf{J} = \int [(\boldsymbol{\rho} \cdot \boldsymbol{\rho})E - \boldsymbol{\rho}\boldsymbol{\rho}] dm \qquad \text{or} \qquad \mathbf{J} = [\hat{\mathbf{x}}_\alpha]^T J [\hat{\mathbf{x}}_\alpha] \tag{6}$$

where $\mathbf{E}$ is a unit tensor and J is the inertia matrix of the system (with respect to the center of mass) expressed in the $\{\hat{\mathbf{x}}_\alpha\}$ basis. The internal angular momentum vector $\mathbf{h}$ is defined as

$$\mathbf{h} = [\hat{\mathbf{x}}_\alpha]^T h = \int \boldsymbol{\rho} \times \boldsymbol{\rho}^* dm \tag{7}$$

The angular momentum then becomes $\mathbf{H} = \mathbf{J} \cdot \boldsymbol{\omega} + \mathbf{h}$ or $\mathbf{H} = [\hat{\mathbf{x}}_\alpha]^T(J\omega + h) = [\hat{\mathbf{x}}_\alpha]^T H$. For free spinning systems, the external torque is equal to zero and the basic relation (4) becomes, $\dot{H} + \tilde{\omega} H = 0$, or $J\dot{\omega} + \tilde{\omega} J \omega + \tilde{\omega} h + \dot{J}\omega + \dot{h} = 0$. The corresponding vectorial relation is $\mathbf{J} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{h} + \mathbf{J}^* \cdot \boldsymbol{\omega} + \mathbf{h}^* = 0$.

The equations of deformation will be the Lagrange equations in the noncyclic variables β_ν

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\beta}_\nu} - \frac{\partial R}{\partial \beta_\nu} + \frac{\partial W}{\partial \dot{\beta}_\nu} = Q_\nu \qquad (\nu = 1 \ldots n)$$

where R is the Routhian function, W the Rayleigh dissipation function, and Q_ν the generalized force for the variable β_ν.

The kinetic energy of rotation of the system is

$$T = \frac{1}{2}\int \dot{\boldsymbol{\rho}} \cdot \dot{\boldsymbol{\rho}} dm$$

Using Eq. (5) and definitions (6) and (7), this energy can be written

$$T = \frac{1}{2}\int \boldsymbol{\rho}^* \cdot \boldsymbol{\rho}^* dm + \boldsymbol{\omega} \cdot \mathbf{h} + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega}$$

or

$$T = \frac{1}{2}\int \dot{\rho}^T \dot{\rho} dm + \omega^T h + \frac{1}{2}\omega^T J \omega$$

As it is assumed that the rotor is symmetric and spins with constant relative angular velocity about its axis of symmetry, the corresponding internal angular momentum is equal to $\mathbf{h}_r = I_{33}'\boldsymbol{\Omega} = I_{33}'\Omega\hat{\mathbf{x}}_3' = [\hat{\mathbf{x}}_\alpha]^T h_r$, where I_{33}' is the moment of inertia about the axis of symmetry.

It can be shown that the internal angular momentum of the system is the sum of $\mathbf{h}_r$ and the internal angular momentum of the corresponding system with frozen rotors, or

$$\mathbf{h} = \int (\mathbf{r} + \mathbf{u}) \times \mathbf{u}^* dm + \mathbf{h}_r \tag{8}$$

The relative time derivative $\mathbf{h}^*$ is then

$$\mathbf{h}^* = \int (\mathbf{r} + \mathbf{u}) \times \mathbf{u}^{**} dm + \mathbf{h}_r^* \tag{9}$$

As the norm of $\mathbf{h}_r$ is constant, $\mathbf{h}_r^*$ is due to the relative angular velocity of the rotor spin axis, $\boldsymbol{\omega}_d$, with respect to the reference axis or, $\mathbf{h}_r^* = \boldsymbol{\omega}_d \times \mathbf{h}_r$. From the constraint equations, it is seen that the integrals appearing in Eq. (8) and (9) do not include linear terms in the deformation variables. It will prove of interest to express $(1/2)\int \boldsymbol{\rho}^* \cdot \boldsymbol{\rho}^* dm$ in terms of integrals defined in the system with frozen rotor, namely,

$$\frac{1}{2}\int \boldsymbol{\rho}^* \cdot \boldsymbol{\rho}^* dm = \frac{1}{2}\int \mathbf{u}^* \cdot \mathbf{u}^* dm + \boldsymbol{\omega}_d \cdot \mathbf{h}_r + \frac{1}{2} I_{33}'\Omega^2$$

It can be seen that the motion of the system is the same as the motion of the system with frozen rotors with an associated internal angular momentum vector $\mathbf{h}_r$ aligned at any time with the rotor axis of symmetry.

We must express the position vector $\boldsymbol{\rho}$ in order to derive the equations of motion. The vector from the platform center of mass to the bearing is denoted $\mathbf{a}$ and remains aligned with $\hat{\mathbf{x}}_3$, thus $\mathbf{a} = a\hat{\mathbf{x}}_3$. Similarly, the vector from the bearing to rotor center of mass is given by $\mathbf{a}' = a'\hat{\mathbf{x}}_3'$.

The vectors from the center of mass to the centers of mass of the platform and the rotor are denoted $\boldsymbol{\rho}_0$ and $\boldsymbol{\rho}_0'$. At equilibrium, these vectors are aligned with $\hat{\mathbf{x}}_3$ and the corresponding distances are denoted d and d'.

From the definition of the center of mass and the fact that $\boldsymbol{\rho}_0' = \boldsymbol{\rho}_0 + \mathbf{a} + \mathbf{a}'$, we have

$$\boldsymbol{\rho}_0 = -d\hat{\mathbf{x}}_3 - [m'/(m + m')][a\hat{\mathbf{x}}_3 + a'\hat{\mathbf{x}}_3' - (a + a')\hat{\mathbf{x}}_3] = [\hat{\mathbf{x}}_\alpha]^T \rho_0$$

and

$$\boldsymbol{\rho}_0' = d'\hat{\mathbf{x}}_3 + [m/(m+m')]\,[a\hat{\mathbf{x}}_3 + a'\hat{\mathbf{x}}_3' - (a+a')\hat{\mathbf{x}}_3] = [\hat{\mathbf{x}}_\alpha]^T \rho_0'$$

where m and m' are the masses of the platform and the rotor respectively. Matrices ρ_0 and ρ_0' are related by

$$m\rho_0 + m'\rho_0' = 0$$

The position vector $\boldsymbol{\rho}$ of an element of mass of the platform has the form $\boldsymbol{\rho} = \boldsymbol{\rho}_0 + \mathbf{x}\ (dm \in P)$, where $\mathbf{x}$ is the position vector of dm with respect to the center of mass of the platform, P being the set of mass elements of the platform. Similarly, the position vector of an element of mass of the rotor can be written as $\boldsymbol{\rho} = \boldsymbol{\rho}_0' + \mathbf{x}'\ (dm \in R)$, where $\mathbf{x}'$ is the position vector of dm with respect to the center of mass of the rotor, R being the corresponding set of mass elements.

The vectors $\mathbf{x}$ and $\mathbf{x}'$ are fixed in the frames $\{\hat{\mathbf{x}}_\alpha\}$ and $\{\hat{\mathbf{x}}_\alpha'\}$, respectively, and can be written as

$$\mathbf{x} = [\hat{\mathbf{x}}_\alpha]^T x = [\hat{\mathbf{x}}_\alpha]^T A^T x, \qquad \mathbf{x}' = [\hat{\mathbf{x}}_\alpha']^T x' = [\hat{\mathbf{x}}_\alpha]^T A^T B^T C^T x'$$

the vectors $\boldsymbol{\rho}$ will then be equal to

$$\boldsymbol{\rho} = \boldsymbol{\rho}_0 + [\hat{\mathbf{x}}_\alpha]^T x = [\hat{\mathbf{x}}_\alpha]^T(\rho_0 + A^T x) \qquad (dm \in P)$$
$$\boldsymbol{\rho} = \boldsymbol{\rho}_0' + [\hat{\mathbf{x}}_\alpha']^T x' = [\hat{\mathbf{x}}_\alpha]^T(\rho_0' + A^T B^T C^T x') \qquad (dm \in R)$$

The corresponding vectors in the deformable body with frozen rotors $\mathbf{p} = \mathbf{r} + \mathbf{u}$, are equal to

$$\mathbf{p} = \boldsymbol{\rho}_0 + [\hat{\mathbf{x}}_\alpha]^T x = [\hat{\mathbf{x}}_\alpha]^T(\rho_0 + A^T x) = \boldsymbol{\rho} \qquad (dm \in P)$$
$$\mathbf{p} = \boldsymbol{\rho}_0' + [\hat{\mathbf{y}}_\alpha]^T x' = [\hat{\mathbf{x}}_\alpha]^T(\rho_0 + A^T B^T x') \qquad (dm \in R)$$

the vectors $\mathbf{r}$ being given by $\mathbf{r} = -d\hat{\mathbf{x}}_3 + [\hat{\mathbf{x}}_\alpha]^T x\ (dm \in P)$ and $\mathbf{r} = d'\hat{\mathbf{x}}_3 + [\hat{\mathbf{x}}_\alpha]^T x'\ (dm \in R)$, where our assumption concerning the body configuration has been used.

The inertia tensors of the platform and the rotor with respect to their centers of mass are given by

$$\mathbf{I} = \int_P (\mathbf{x}\cdot\mathbf{x})\mathbf{E} - \mathbf{x}\mathbf{x}dm = [\hat{\mathbf{x}}_\alpha]^T I [\hat{\mathbf{x}}_\alpha]$$

$$\mathbf{I}' = \int_R (\mathbf{x}'\cdot\mathbf{x}')\mathbf{E} - \mathbf{x}'\mathbf{x}'dm = [\hat{\mathbf{x}}_\alpha']^T I' [\hat{\mathbf{x}}_\alpha'] = [\hat{\mathbf{y}}_\alpha]^T I' [\hat{\mathbf{y}}_\alpha]$$

where matrices I and I' are written in the form

$$I = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{11} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}, \qquad I' = \begin{bmatrix} I_{11}' & 0 & 0 \\ 0 & I_{11}' & 0 \\ 0 & 0 & I_{33}' \end{bmatrix}$$

The inertia tensor of the system $\mathbf{J}$ is given by

$$\mathbf{J} = \int (\boldsymbol{\rho}\cdot\boldsymbol{\rho})E - \boldsymbol{\rho}\boldsymbol{\rho}dm = \frac{m}{m'}(m+m')[(\boldsymbol{\rho}_0\cdot\boldsymbol{\rho}_0)\mathbf{E} - \boldsymbol{\rho}_0\boldsymbol{\rho}_0] + \mathbf{I} + \mathbf{I}'$$

At equilibrium, the transverse and axial moments of inertia, written I and I_3, respectively, are equal to

$$I = I_{11} + I_{11}' + (m/m')(m+m')d^2, \qquad I_3 = I_{33} + I_{33}'$$

Using the above definitions and the linearized form of the transformation matrices between body frames, the linearized constraint Eqs. (3) are

$$I_{11}\gamma_1 + I_{11}'(\beta_1 + \gamma_1) + md[a\gamma_1 + a'(\gamma_1 + \beta_1)] = 0$$
$$I_{11}\gamma_2 + I_{11}'(\beta_2 + \gamma_2) + md[a\gamma_2 + a'(\gamma_2 + \beta_2)] = 0$$
$$(I_{33} + I_{33}')\gamma_3 = 0$$

The expression of the γ variables in terms of β_1 and β_2 are then $\gamma_1 = \rho_0\beta_1$, $\gamma_2 = \rho_0\beta_2$, $\gamma_3 = 0$, where $\rho_0 = -(I_{11}' + Mda')/I$.

Within the desired approximation, the inertia matrix J can then be written

$$J_{11} = J_{22} = I, \qquad J_{31} = J_{13} = 2\Lambda_{31}\beta_2$$
$$J_{32} = J_{23} = +2\Lambda_{32}\beta_1, \qquad J_{33} = I_3 + \Gamma_{33}(\beta_1^2 + \beta_2^2)$$

where

$$2\Lambda_{31} = -2\Lambda_{32} = I_3\rho_0 + I_{33}'$$

$$\Gamma_{33} = (I_{11} - I_{33})\rho_0^2 + (I_{11}' - I_{33}')(1+\rho_0)^2 + \frac{mm'}{m+m'}\times[a\rho_0 + a'(1+\rho_0)]^2$$

The nominal value of ω_3 being ω_0, the linearized angular momentum matrix H is equal to

$$H = \begin{bmatrix} I\omega_1 + [2\Lambda_{31}\omega_0 + I_{33}'\Omega(1+\rho_0)]\beta_2 \\ I\omega_2 - [2\Lambda_{31}\omega_0 + I_{33}'\Omega(1+\rho_0)]\beta_1 \\ I_3\omega_3 + I_{33}'\Omega \end{bmatrix} \tag{10}$$

and the linearized Euler equations are then

$$I\dot{\omega}_1 + [(I_3 - I)\omega_0 + I_{33}'\Omega]\omega_2 + [2\Lambda_{31}\omega_0 + I_{33}'\Omega(1+\rho_0)](\dot{\beta}_2 + \omega_0\beta_1) = 0$$

$$I\dot{\omega}_2 - [(I_3 - I)\omega_0 + I_{33}'\Omega]\omega_1 - [2\Lambda_{31}\omega_0 + I_{33}'\Omega(1+\rho_0)](\dot{\beta}_1 - \omega_0\beta_2) = 0$$

$$I_3\dot{\omega}_3 = 0$$

Retaining only quadratic terms, the integral $\frac{1}{2}\int \mathbf{u}^*\cdot\mathbf{u}^* dm$ can be written as

$$\tfrac{1}{2}\int \mathbf{u}^*\cdot\mathbf{u}^* dm = \tfrac{1}{2}m_0(\dot{\beta}_1^2 + \dot{\beta}_2^2)$$

where

$$m_0 = I_{11}\rho_0^2 + I_{11}'(1+\rho_0)^2 + \frac{mm'}{m+m'}[a\rho_0 + a'(1+\rho_0)]^2$$

The quadratic terms of $\boldsymbol{\omega}_d\cdot\mathbf{h}_r$ are

$$\boldsymbol{\omega}_d\cdot\mathbf{h}_r = I_{33}'\Omega[(1+\rho_0+\rho_0^2)\dot{\beta}_1\beta_2 - \rho_0\beta_1\dot{\beta}_2]$$

Similarly, up to the quadratic terms, the following matrices may be written

$$\boldsymbol{\omega}\cdot\int(\mathbf{r}+\mathbf{u})\times\mathbf{u}^* dm = \tfrac{1}{2}\Gamma\omega_0(\dot{\beta}_1\beta_2 - \dot{\beta}_2\beta_1)$$

and

$$\boldsymbol{\omega}\cdot\mathbf{h}_r = I_{33}'\Omega(1+\rho_0)[\omega_1\beta_2 - \omega_2\beta_1] - \frac{I_{33}'\Omega}{2}(1+\rho_0)^2\times(\beta_1^2 + \beta_2^2)\omega_0$$

where

$$\Gamma = (I_{33} - 2I_{11})\rho_0^2 + (I_{33}' - 2I_{11}')(1+\rho_0)^2 - 2\frac{mm'}{m+m}\times[a\rho_0 + a'(1+\rho_0)]^2$$

We will assume a linear viscoelastic bearing assembly, so that the potential energy of deformation is

$$U = \tfrac{1}{2}k(\beta_1^2 + \beta_2^2)$$

where k is the stiffness constant of the two parts together. Even if the deflection is supposed to be the same for both parts of the bearing, the deformation rate is different for the platform and the rotor parts. If damping is modelled by two orthogonal dampers (with damping coefficient c), associated with the platform part of the bearing assembly and two orthogonal dampers (with damping coefficient c'), associated with the rotor part, the Rayleigh dissipation function is

$$W = (c/2)(\dot{\beta}_1^2 + \dot{\beta}_2^2) + (c'/2)[(\dot{\beta}_1 + \Omega\beta_2)^2 + (\dot{\beta}_2 - \Omega\beta_1)^2]$$

The Lagrange equations in the variables β_1 and β_2 are then

$$m_0\ddot{\beta}_1 + [2\Lambda_{31}\omega_0 + I_{33}'\Omega(1+\rho_0)]\dot{\omega}_2 + [\Gamma\omega_0 + I_{33}'\Omega(1+\rho_0)^2]\dot{\beta}_2 + \pi\beta_1 + (c+c')\dot{\beta}_1 + c'\Omega\beta_2 = 0$$

$$m_0\ddot{\beta}_2 - [2\Lambda_{31}\omega_0 + I_{33}'\Omega(1+\rho_0)]\dot{\omega}_1 - [\Gamma\omega_0 + I_{33}'\Omega(1+\rho_0)^2]\dot{\beta}_1 + \pi\beta_2 + (c+c')\dot{\beta}_2 - c'\Omega\beta_1 = 0$$

where

$$\pi = \mathrm{K} - \Gamma_{33}\omega_0{}^2 + I_{33}{}'\Omega\omega_0(1+\rho_0)^2$$

If the variables ω_1 and ω_2 are replaced by expressions (1), the equations of motion can be written in the form

$$A'\ddot{x} + G\dot{x} + \mathrm{K}x + D\dot{x} + Fx = 0 \tag{11}$$

where the matrix x is the matrix $x = [\theta_1\ \theta_2\ \beta_1\ \beta_2]^T$, and A' K and D are the symmetric matrices

and will be written

$$\dot{\theta}_1 = -\Lambda\theta_2 - A\beta_2, \qquad \dot{\theta}_2 = \Lambda\theta_1 + A\beta_1 \tag{13}$$

where

$$\Lambda = [(I_3 - I)\omega_0 + I_{33}{}'\Omega]/I,$$
$$A = [2\Lambda_{31}\omega_0 + I_{33}{}'\Omega(1+\rho_0)]/I$$

A new testing function V' is obtained by combining V and the components of **H** along $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$. The time derivative of

$$A' = \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & m_0 & 0 \\ 0 & 0 & 0 & m_0 \end{array}\right], \qquad D = (c+c')\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathrm{K} = \begin{bmatrix} (I_3 - I)\omega_0{}^2 + I_{33}{}'\Omega\omega_0 & 0 & 2\Lambda_{31}\omega_0{}^2 + I_{33}{}'\Omega(1+\rho_0)\omega_0 & 0 \\ 0 & (I_3 - I)\omega_0{}^2 + I_{33}{}'\Omega\omega_0 & 0 & 2\Lambda_{31}\omega_0{}^2 + I_{33}{}'\Omega(1+\rho_0)\omega_0 \\ 2\Lambda_{31}\omega_0{}^2 + I_{33}{}'\Omega(1+\rho_0)\omega_0 & 0 & \pi & 0 \\ 0 & 2\Lambda_{31}\omega_0{}^2 + I_{33}{}'\Omega(1+\rho_0)\omega_0 & 0 & \pi \end{bmatrix}$$

the matrices G and F are skew symmetric matrices,

$$G = \begin{bmatrix} 0 & (I_3 - 2I)\omega_0 + I_{33}{}'\Omega & 0 & 2\Lambda_{31}\omega_0 + I_{33}{}'\Omega(1+\rho_0) \\ -(I_3 - 2I)\omega_0 - I_{33}{}'\Omega & 0 & -2\Lambda_{31}\omega_0 - I_{33}{}'\Omega(1+\rho_0) & 0 \\ 0 & 2\Lambda_{31}\omega_0 + I_{33}{}'\Omega(1+\rho_0) & 0 & \Gamma\omega_0 + I_{33}{}'\Omega(1+\rho_0)^2 \\ -2\Lambda_{31}\omega_0 - I_{33}{}'\Omega(1+\rho_0) & 0 & -\Gamma\omega_0 - I_{33}{}'\Omega(1+\rho_0)^2 & 0 \end{bmatrix}$$

$$F = c'\Omega\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

The matrix F can be written $F = DS$ and a candidate S matrix is

$$S = \frac{c'}{c+c'}\Omega\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Stability Analysis

The stability of the linear system (11) will be determined by use of the Liapunov second method as the form of the equations permits a rather simple construction of a testing function.

In fact, as matrices S, $A'S$ and KS are skew symmetric, and matrix GS is symmetric, we can consider

$$V = \tfrac{1}{2}\dot{x}^T A\dot{x} - x^T S A'\dot{x} + \tfrac{1}{2}x^T(\mathrm{K} - GS)x$$

as a testing function. The time derivative of V along the trajectory of the system is simply $\dot{V} = -(\dot{x} + Sx)^T C(\dot{x} + Sx)$. This derivative is negative semidefinite and, when damping is complete, the positive definiteness of V is a necessary and sufficient condition for asymptotic stability of the linearized system.

It must be noted that when the linearized system is asymptotically stable, the corresponding nonlinear system is also stable, the nonlinear terms being terms of higher order in the variables.

Here this condition is only sufficient for (neutral) stability as the damping is not complete. In fact, $\omega_1 = \omega_2 = \beta_1 = \beta_2 = 0$ is a solution of the system and any θ_1 and θ_2 satisfying the equations

$$\dot{\theta}_1 - \omega_0\theta_2 = 0, \qquad \dot{\theta}_2 + \omega_0\theta_1 = 0 \tag{12}$$

provides a nontrivial solution of the system. No statement can be made concerning the stability of the nonlinear system when V is used.

Nonzero solutions of (12) are obtained when the initial conditions introduce a change in the total angular momentum. If we exclude this possibility, the angular momentum remains constant and as $\hat{\mathbf{a}}_3$ is aligned with **H**, the components of **H** along $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ must be identically equal to zero. In linear approximation, these conditions on the state variables are

$$H_1 + \theta_2 H_3 = 0, \qquad H_2 - \theta_1 H_3 = 0$$

V' is equal to the time derivative of V, but now there is no trajectory different from the trivial solution along which $\dot{V}'$ is identically equal to zero (except perhaps for particular values of the system parameters for which damping is not complete).

The positive definiteness of a function being determined in the neighborhood of the origin by its quadratic terms, the function V will be combined with the linearized relations (13) in order to obtain the Liapunov function V'. This combination will be chosen such that the function V' be independent of $\dot{\theta}_1$ and $\dot{\theta}_2$.

Defining the vector $y = [\dot{\beta}_1\dot{\beta}_2\theta_1\theta_2\beta_1\beta_2]$, the function V' will be written under the form

$$V' = \tfrac{1}{2}y^T My$$

with

$$M_{11} = M_{22} = m_0$$

$$M_{33} = M_{44} = W_1 + I\Lambda^2 - 2\frac{c'}{c+c'}I\Omega\Lambda$$

$$M_{55} = M_{66} = W_2 + IA^2$$

$$M_{16} = M_{61} = -M_{25} = -M_{52} = [(c'/(c+c')]\, m_0\Omega$$

$$M_{35} = M_{53} = M_{46} = M_{64} = W_3 + IA\Lambda - [(c'/(c+c')]I\Omega A$$

where

$$W_1 = (I_3 - I)\omega_0{}^2 + I_{33}{}'\Omega\omega_0 + \frac{c'}{c+c'}\Omega[(I_3 - 2I)\omega_0 + I_{33}{}'\Omega]$$

$$W_2 = \pi + \frac{c'}{c+c'}\Omega[\Gamma\omega_0 + I_{33}{}'\Omega(1+\rho_0)^2]$$

$$W_3 = [2\Lambda_{31}\omega_0 + I_{33}{}'\Omega(1+\rho_0)]\left[\omega_0 + \frac{c'}{c+c'}\Omega\right]$$

When V' is positive definite, the linear system is partially asymptotically stable in the variable y. From Eq. (13), the variables $\dot{\theta}_1$ and $\dot{\theta}_2$ must then tend to zero and it can be concluded that the linearized (and the corresponding nonlinear) system is asymptotically stable.

The function V' is positive definite when the matrix M is positive definite. From Sylvester's criterion, the determinants of all the principal minors of M have to be positive to have asymptotic stability or

$$W_1 + I\Lambda\left(\Lambda - 2\frac{c'}{c+c'}\Omega\right) > 0$$

$$\left[W_1 + I\Lambda\left(\Lambda - 2\frac{c'}{c+c'}\Omega\right)\right]\left[W_2 + IA^2 - \left(\frac{c'}{c+c'}\right)^2 m_0\Omega^2\right] - \left[W_3 + IA\left(\Lambda - \frac{c'}{c+c'}\Omega\right)\right]^2 > 0, \qquad m_0 > 0$$

The system becomes unstable as soon as one of the inequalities is reversed. When one of the above relations is an equality no conclusion can be drawn concerning the stability of the system. Ignoring the last restriction, the above conditions are necessary and sufficient.

The first inequality is a function of the inertia configuration, the spin rates and of the ratio $c'/(c+c')$. The second inequality is a function of the above parameters and of the stiffness of the bearing and it permits to determine the minimum value of this parameter. The last condition is generally satisfied, m_0 being the generalized mass associated with the deformation variables.

If we assume that the internal stiffness is large enough to maintain "internal stability", i.e., to satisfy the second inequality, the first condition can be considered as the critical stability condition.

Recalling the definition of the various parameters, this condition can be written

$$(I_3\omega_0 + I_{33}'\Omega)\left[(I_3 - I)\omega_0 + \left(I_{33}' - \frac{c'}{c+c'}I\right)\Omega\right] > 0 \qquad (14)$$

Now let us recall that, for the symmetrical system under consideration, we arbitrarily took one body as the platform, the other being the rotor. If we now choose the second body as the platform, we have to substitute, $\omega_0 + \Omega$ for ω_0, $-\Omega$ for Ω, $I_3 - I_{33}'$ for I_{33}', and $1 - c'/(c+c')$ for $c/(c+c')$, and as could be expected here, the inequality (14) remains invariant under this transformation.

When $c' = 0$, we have the well-known stability condition for a deformable gyrostat in free space, derived from the Routhian analysis or by Liapunov method for general deformation in.[9]

The above criteria were obtained for a system with $\omega_0 = 0$, by Mingori.[10] The stability condition (14) then reduces to $I_{33}' - c'/(c+c')I > 0$. This result is also obtained from the energy sink criterion[11] when the deformations are small enough to maintain the natural frequency of the system close to the nutation frequency of the equivalent rigid gyrostat. In fact, the variables β_1 and β_2 are then almost periodic with $\Lambda = I_{33}'\Omega/I$. It can then be seen that the average energy dissipation in the platform is proportional to $c\Lambda^2$ and that the average energy dissipation in the rotor is proportional to $c'(\Lambda - \Omega)^2$.

A stability criterion similar to (14) has been obtained by Scher[12] who considers a symmetric dual-spin spacecraft with a flexible bushing tied to neither the rotor nor the platform, but which rotates at a constant rate $r\Omega$ with respect to the platform. When $r = 0$, the bushing is tied to the platform, and when $r = 1$, it is tied to the rotor. The corresponding stability criterion, obtained from energy sink consideration, is

$$(I_3\omega + I_{33}'\Omega)[(I_3 - I)\omega + (I_{33}' - rI)\Omega] > 0$$

By comparison with the result of this paper, it can be concluded that Scher's apportioning factor r must be taken equal to $c'/(c+c')$ for a viscoelastic bearing assembly with energy dissipation in both parts.

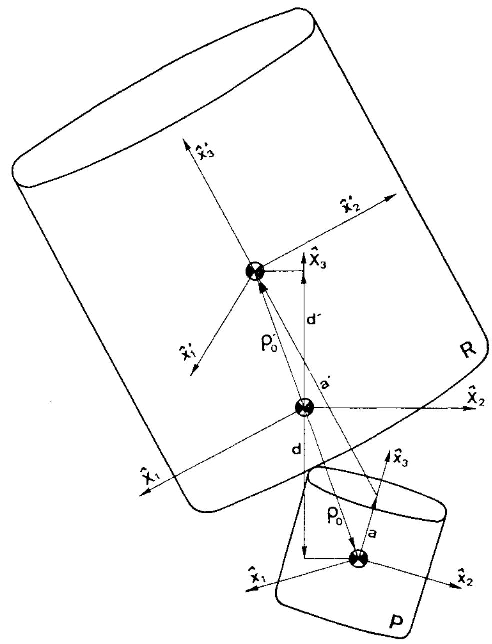

Fig. 1 Dual-spin spacecraft with flexible bearing assembly.

Similarly, when $\Omega = 0$, we have the stability condition for a deformable body in free space ($I_3 > I_1$) provided the stiffness is large enough.

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